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# Log-volatility enhanced GARCH models for single asset returns

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## Abstract

This paper presents an alternative approach to modelling and forecasting single asset return volatility. A new, flexible framework is proposed, one which may be considered a development of single-equation GARCH-type models. In this approach an additional equation is added, which binds logarithms of conditional volatility and observed volatility, as measured by the Garman-Klass variance estimator. It enables more information to be retrieved from data. Proposed models are compared with benchmark GARCH and range-based GARCH (RGARCH) models in terms of prediction accuracy. All models are estimated with the maximum likelihood method, using time series of EUR/PLN, EUR/USD, EUR/GBP spot rates quotations as well as WIG20, Dow Jones industrial and DAX indexes. Results are encouraging, especially for foreasting *Value-at-Risk*. Log-volatility enhanced models achieved lesser rates of VaR exception, as well as lower coverage test statistics, without being more conservative than their single-equation counterparts, as their forecast error measures are to some degree similar.

**Keywords:** GARCH, range-based volatility estimators, observed volatility, *Value-at-Risk*, volatility forecasting

**JEL:** C13, C32, C53, C58, G10, G17

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# 1. Introduction

Volatility modeling is at the forefront of financial econometric interest. The increasing importance of this subject comes from both business and regulatory institutions in the financial market sector. Over the past three decades, dozens of models have been proposed. All of them address specific challenges of volatility modeling including leptokurtosis of empirical returns distribution, volatility clustering, and the asymmetry effect. There is a common belief that, at least to a certain degree, volatility is predictable. Models built to forecast volatility are called conditional volatility models because they try to infer future volatility conditional on present information set. This paper focuses on the group of volatility models based on generalized autoregressive conditional heteroskedasticity and proposes a new model built around well known GARCH-type models. The aim is to incorporate more information into classical GARCH framework by adding an additional equation. This equation binds logarithms of conditional variance and observed volatility measured by some kind of variance estimator. The proposed flexible framework not only enhances the forecasting performance of GARCH-type models, but also allows some conclusions to be drawn about the relationship between asset returns and their observed volatility. One of the features of this new approach is focusing on joint distribution of returns and their observed volatility. This is possible due to the use of more efficient range-based daily variance estimators instead of squared returns (or errors) as a volatility proxy.

In this paper six financial time series are investigated: EUR/PLN, EUR/USD and EUR/GBP spot rates quotations as well as WIG20, Dow Jones industrial and DAX indexes. Four conditional volatility models are employed to obtain volatility predictions. These are: a well-known GARCH model and its range-based counterpart (RGARCH), as well as two newly developed log-volatility enhanced models derived from GARCH and RGARCH models, respectively. Log-volatility enhanced models show very promising performances especially in terms of forecasting *Value-at-Risk*. Moreover, they allow simultaneous dependencies between observed volatility and returns to be examined.

The rest of the paper is organized as follows. Section 2 reviews volatility estimators based on high, low, open, and close prices (range-based estimators) and briefly describes volatility models that are the most relevant from this paper's point of view. Section 3 contains derivations of the proposed models. In section 4, empirical results are presented for both in-sample and out-of-sample analyses. Section 5 concludes.

# 2. Literature review

In the literature there are several classes of volatility models. However two of them are arguably the most popular for modelling volatility with daily data; those are stochastic volatility and (generalized) autoregressive conditional heteroskedasticity models. The main difference between SV and GARCH models is an assumption about the nature of volatility: in the case of GARCH-type models, volatility is considered a deterministic process; in the case of SV models, volatility has a fully stochastic nature. Despite its conceptual attractiveness, stochastic volatility models are not as popular as their GARCH--type counterparts. The main reason for this is the fact that SV models are, in general, computationally demanding, as their likelihood can not be obtained in a closed form. A detailed overview of SV models is provided by Shephard and Andersen (2009), while estimation techniques are described, e.g. in Broto and Ruiz (2004). Depth review of GARCH-type models can be found in Terasvirta (2009); an interesing paper of Bollerslev (2008) provides a glossary to ARCH/GARCH models.

In the classical framework, both SV and GARCH-type models only demand time series of asset close prices. Recently, models using extreme value volatility estimators (the so called range-based estimators) are becoming increasingly popular. Before some "range-based" volatility models are discussed, a brief review of extreme value volatility estimators is warranted.

Let $S_t$, the price of the asset, follow geometric Brownian motion, thus satisfy the following condition:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

where $W_t$ is a Wiener process, and $\mu$ and $\sigma$ denote drift and diffusion coeffcients.

The solution of stochastic differential equation given by equation (1) is:

$$\ln S_t = \ln S_0 + (\mu - \frac{\sigma^2}{2})t + \sigma W_t \tag{2}$$

Suppose that there are $N$ observations of daily data and let $C_n$, $O_n$, $H_n$ and $L_n$ be respectively, the close, the open, the highest and the lowest price on day $n$. Then the classical close-to-close estimator of $\sigma^2$ is given by:

$$\hat{\sigma}^2_{CC} = \frac{1}{N-1} \sum_{n=1}^{N} (\ln C_n / C_{n-1} - \frac{1}{N} \sum_{n=1}^{N} \ln C_n / C_{n-1})^2 \tag{3}$$

Thus, the classical estimator of $\sigma^2$ is a sample variance of logarithmic returns. It is possible to simplify formula (3) by assuming that:

$$\mu = \frac{\sigma^2}{2} \tag{4}$$

Then $\ln S_t$ follows continuous random walk without drift and (3) reduces to:

$$\hat{\sigma}^2_{CC} = \frac{1}{N} \sum_{n=1}^{N} (\ln C_n - \ln C_{n-1})^2 \tag{5}$$

Under assumptions (1) and (4), Parkinson (1980) introduced an alternative estimator of $\sigma^2$:

$$\hat{\sigma}^2_{Park} = \frac{1}{4N \ln 2} \sum_{n=1}^{N} (\ln H_n - \ln L_n)^2 \tag{6}$$

The expression: $(\ln H - \ln L)$ is often called "range", thus extreme value variance estimators are also described as "range-based" estimators. Parkinson estimator is simply a mean of squared ranges times constant.

Defining a relative efficiency of estimators as a ratio $\mathrm{Var}(\sigma^2)/\mathrm{Var}(\sigma^2_{Park})$, it can be shown that the Parkinson estimator is up to 4.9 times more efficient than the classical variance estimator (which means that the Parkinson estimator has 4.9 times lower variance than the classical close-to-close estimator). Garman and Klass (1980) proposed an even more efficient variance estimator, one that uses not only the highest and the lowest, but also close and open prices:

$$\hat{\sigma}^2_{GK} = \frac{1}{N}\sum_{n=1}^{N} 0.5(\ln(H_n/L_n))^2 - (2\ln 2 - 1)(\ln(C_n/O_n))^2 \tag{7}$$

The theoretical relative efficiency of the Garman-Klass estimator is 7.4, but similiarly to Parkinson's, it is derived under the assumption that the logarithm of asset price follows a continuous random walk without drift. Rogers and Satchell (1991) removed this assumption and derived an estimator that is robust to drift in a log-price process:

$$\begin{aligned}\hat{\sigma}^2_{RS} &= \frac{1}{N}\sum_{n=1}^{N} \ln(H_n/O_n)(\ln(H_n/O_n) - \ln(C_n/O_n)) \\ &+ \frac{1}{N}\sum_{n=1}^{N} \ln(L_n/O_n)(\ln(L_n/O_n) - \ln(C_n/O_n))\end{aligned} \tag{8}$$

It should be underlined that in the case of daily variance, $N = 1$ for all described estimators. There are a few other range-based estimators, detailed overview of extreme value volatility estimators is presented e.g. in Li and Weinbaum (2000).

The common drawback of range-based volatility estimators is their downward bias, which is reported by Garman and Klass (1980), Beckers (1983) and Wiggins (1991). There are two sources of aforementioned bias: the periods when markets are closed, and the discrete nature of observed prices. Due to both of them, the observed highest and lowest daily prices are respectively lower and higher than the true ones.

The pioneering research using range-based estimators in volatility modelling was conducted by Alizadeh, Brandt and Diebold (2001). In their paper, a range-based stochastic volatility model was proposed. Authors found a useful distributional property of range – they argue that logarithm of range is approximately Gaussian. This improves the performance of the QMLE (quasi-maximum likelihood estimation) method of SV models estimation. A different approach was chosen by Chou (2005). He examined the dynamic behaviour of range and formulated a conditional autoregressive range (CARR) model. In using the CARR model, conditional volatility is obatined in two steps: first, a conditional range is predicted, then forecast of volatility is computed by inserting conditional range into Parkinson's formula (6). The first attempt to incoporate extreme value volatility estimators into GARCH framework was made by Brandt and Jones (2006) as they proposed REGARCH model (range--based exponential GARCH). Authors used the aforementioned distributional property of log-range and reformulated conditional variance equation of EGARCH by replacing absolute value of return with logarithm of range. A different approach was chosen by Lildholdt (2003). The author leaves the conditional variance equation unchanged in comparison to the classic GARCH(1,1) model, but estimates model parameters using the joint distribution of the vector of maximal, minimal and close

(HLC) prices. The exact formula for density function of HLC prices distribution is complicated and would not be presented in this paper. Moreover, it contains an infinite sum, thus requires truncation and may be difficult to implement. Recently, an extension of the model proposed by Lildhold has been developed by Fiszeder and Perczak (2013). Authors not only use the joint distribution of HLC prices, but also modify the conditional variance equation by inserting a custom range-based variance estimator in place of squared innovations. Arguably the simpliest model that incoporates range-based estimators into GARCH framework is the RGARCH (range-based GARCH) model. The main assumption of this model is that squared errors in conditional variance equation can be replaced with a more efficient volatility estimator. The exact formula of the RGARCH(1,1) model is very similar to the GARCH(1,1) model and can be expressed in a following way:

$$\begin{aligned} r_t &= \mu_t + \varepsilon_t \\ \varepsilon_t &\sim N(0, h_t) \\ h_t &= \varpi + \alpha \hat{\sigma}_{t-1}^2 + \beta h_{t-1} \end{aligned} \tag{9}$$

where $r_t$ is a return, $\mu_t$ is a potentialy time-varying mean of return, and $\varepsilon_t$ is an error with zero mean and conditional variance $h_t$ that depends on its past values and some range-based variance estimator $\hat{\sigma}_t^2$.

The only difference between the classical GARCH(1,1) and RGARCH (1,1) is the specification of conditional variance equation. Several versions of RGARCH models that differ in the variance estimator ($\hat{\sigma}^2$) used are presented in Molnar (2011), and Skoczylas (2013). The main drawback of the RGARCH model is that the unconditional variance of $\varepsilon$ cannot be calculated using parameter estimates due to the fact that range-based volatility estimators are downward biased and in general:

$$E\hat{\sigma}_t^2 \neq Eh_t \tag{10}$$

## 3. Models' derivation

In the classical GARCH framework, returns are assumed to be normally distributed with a conditional mean $\mu_t$ and a conditional variance $h_t$. The conditional mean is often modelled as an ARMA process; however, for simplicity a constant mean is assumed in this paper:

$$\begin{aligned} r_t &= \mu + \varepsilon_t \\ \varepsilon_t &\sim N(0, h_t) \end{aligned} \tag{11}$$

To completely describe a GARCH-type model, one has to specify the conditional variance equation. In this paper two different kinds of conditional variance equations are used. The first one stems from the standard GARCH(1,1) model:

$$h_t = \varpi + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1} \tag{12}$$

while the second one comes from the aforementioned RGARCH model and employs the Garman-Klass estimator (described by formula 7):

$$h_t = \varpi + \alpha\hat{\sigma}^2_{GK,t-1} + \beta h_{t-1} \tag{13}$$

As it was pointed out in section 2, using extreme value estimators one can measure daily variance of returns. When treated as an observed volatility, it is a source of additional information. Before we incorporate it into the model framework, it is necessary to make some assumptions on that observed volatility. Certainly, the observed volatility (measured by Garman-Klass estimator) is a noisy approximation of true volatility of returns (conditional variance). Moreover, Garman-Klass estimator tends to be downward biased. Taking that into account, it is possible to obtain a relationship between observed volatility and conditional variance of $\varepsilon$:

$$\begin{aligned} \hat{\sigma}^2_{GK,t} &= Kh_t\xi_t \\ \xi_t &\sim \ln N(0,v) \\ E\xi_t &= \exp(v/2) \end{aligned} \tag{14}$$

where constant $K$ is included to capture the potential bias in Garman-Klass estimator, and is expected to be lower than $(\exp(v/2))^{-1}$ (due to downward bias of extreme value variance estimators). $\xi_t$ – a random error – is distributed log-normally with location parameter 0, and scale parameter $\sqrt{v}$.

Equation (14) may be treated as some kind of additional restriction imposed on conditional variance $h_t$ that should improve precision of parameter estimation.

Taking logarithms of both sides of (14) leads to:

$$\ln\hat{\sigma}^2_{GK,t} = k + \ln h_t + \eta_t \tag{15}$$

where $\eta_t = \ln\xi_t$ has Gaussian distribution with zero mean and variance $v$.

The next step is to investigate the joint distribution of $\varepsilon$ and $\eta$. Since they are both normally distributed with zero mean, their joint distribution is fully described by their covariance matrix:

$$\begin{aligned} \begin{bmatrix} \varepsilon_t \\ \eta_t \end{bmatrix} &\sim N(\mathbf{0}, \Omega_t) \\ \Omega_t &= \begin{bmatrix} h_t & \rho\sqrt{h_t v} \\ \rho\sqrt{h_t v} & v \end{bmatrix} \end{aligned} \tag{16}$$

A correlation $\rho$ between $\varepsilon$ and $\eta$ is set to be constant.

The first model that uses the conditional variance equation coming from the standard GARCH model will be called LVE-GARCH (log-volatility enhanced GARCH), whereas the second model, based on RGARCH will be called LVE-RGARCH.

Proposed models rely on bivariate normal distribution mostly due to its desired properties (straightforward relationship between normal and log-normal distribution, intuitive interpretation of parameteres, etc.) as well as relative simplicity of estimation. However this assumption may cause some drawbacks in comparison with GARCH-type models that rely on leptokurtic distributions, thus dual-equation models with joint distribution different than bivariate normal should be examined in future research.

There exists one theoretical advantage of the LVE-RGARCH model over RGARCH. In the LVE-RGARCH model, it is possible to obtain a formula for unconditional variance. As it was mentioned before, in the single equation RGARCH model, unconditional variance cannot be calculated using parameter estimates, but in the LVE-RGARCH model, using properties of log-normal distribution and equation (15) it could be shown that:

$$E\hat{\sigma}^2_{GK,t} = EKh_t\xi_t = Kh_tE\xi_t = K\exp(v/2)h_t = \exp(k + v/2)h_t \tag{17}$$

Thus, unconditional variance in a LVE-RGARCH model may be expressed as:

$$\mathrm{Var}(\varepsilon_t) = \frac{\varpi}{1 - \exp(k + v/2)\alpha - \beta} \tag{18}$$

Now it follows that sufficient conditions for covariance stationarity of $\varepsilon$ are:

$$\varpi > 0, \alpha \geq 0, \beta \geq 0 \tag{19.1}$$

$$\exp(k + v/2)\alpha + \beta < 1 \tag{19.2}$$

It should be underlined that such an inference was possible due to the bivariate nature of the LVE-RGARCH model, and could not be conducted in a single equation RGARCH model.

Using the well-known properties of bivariate normal distribution it is possible to determine conditional distribution of $\eta$ given $\varepsilon$:

$$\eta_t \mid \varepsilon_t \sim N(\rho\varepsilon_t\sqrt{\frac{v}{h_t}}, (1-\rho^2)v) \tag{20}$$

A sign of conditional expectation of $\eta_t$ given $\varepsilon_t$ depends solely on the signs of $\varepsilon_t$ and $\rho$. Knowing $\rho$, one can find how present returns affect present observed volatility. In equities and securities markets, increased volatility usually occurs during periods of downward trend, thus one should expect negative values of $\rho$. It is less clear in the case of foreign exchange markets, where the convention of quotation plays a crucial role. Generally, if the base currency is a currency of a developed economy and the counter currency is a currency of an emerging market, the pair rates tend to follow an upward trend during turbulent periods; in such a case one should expect positive values of $\rho$. The opposite conclusion holds when a reverse relation between currencies occurs. It is hard to predict the sign of $\rho$ when both currencies are currencies of developed economies or emerging markets.

All four analysed models are estimated using the quasi maximum likelihood method. In the case of GARCH and RGARCH models, the log-likelihood function is well-known and will not be presented here. In the case of newly developed log-volatility enhanced models the log-likelihood function is obtained using the properties of bivariate normal distribution and may be expressed in the following way:

$$
\begin{aligned}
L(r_t, \ln \hat{\sigma}_t^2; \theta) &= -N \ln 2\pi - 0.5 \sum_{t=1}^{N} \ln |\Omega_t| - 0.5 \sum_{t=1}^{N} y_t^T \Omega_t^{-1} y_t \\
y_t &= \begin{bmatrix} r_t - \mu \\ \ln \hat{\sigma}_{GK,t}^2 - k - \ln h_t \end{bmatrix} \\
\theta &= [\mu, \varpi, \alpha, \beta, \rho, v, k]^T
\end{aligned} \tag{21}
$$

where $\theta$ is a vector of parameters to be estimated, $\Omega_t$ is given by equation (16) and $N$ is the number of observations.

Log-likelihood functions were maximized using the "nlminb" routine from R package stats. The starting values were fixed for both the LVE-GARCH and the LVE-RGARCH for all the analysed time series as equal to $\theta$ = [0, 0.001, 0.2, 0.7, 0, 0.1, -0.2]. The standard errors of parameter estimates were calculated using the "Huber sandwich estimator".

To find whether additional information stemming from equation (14) in fact improves the precision of estimation of conditional variance equation parameters, numerical simulations were conducted. The analysed models were divided into two group: the first one with GARCH and LVE-GARCH, and the second one with RGARCH and LVE-RGARCH. For each group 100 artificial time series of 3000 daily observations of high, low, open and close prices were generated. In both cases random walk without drift were assumed as a data generating process with random error following GARCH(1,1) process for first and RGARCH(1,1) process for second group. 30 000 daily price changes was assumed, so an asset price $P$ in day $t$ and moment $i$ ($i \in \{1, ..., 30\,000\}$) can be expressed in the following way:

$$
\begin{aligned}
P_{t,i} &= P_{t,i-1} \exp(\sqrt{h_t}\, \varepsilon_{t,i}) \\
\varepsilon_{t,i} &\sim N(0{,}1/30\,000)
\end{aligned} \tag{22}
$$

where the conditional variance $h_t$ is described by either the GARCH(1,1) or the RGARCH(1,1) process with parameters set to: $\varpi$ = 0.00001, $\alpha$ = 0.1, $\beta$ = 0.85.

Each time, additionally, the LVE-GARCH and the LVE-RGARCH models with different values of starting parameters ($\theta$ = [0, 0.1, 0.6, 0.1, 0, 0.5, 0]) were estimated, to find out whether the proposed models are sensitive to starting paremeter values. For all four models MAPE (mean absolute percentage error) was calculated with the use of the actual values of daily variance. Table 1 presents the results of simulations. Means of parameter estimates along with their standard deviations are shown. Models with the alternative vector of starting parameter values are marked with an asterisk. In both cases parameter estimates coming from log-volatility enhanced models are not only closer to real values than those coming from single-equation models, but they are also more precise, as their standard deviations are lower. Log-volatility enhanced models also lead to lower values of the MAPE loss function. Moreover, it should be underlined that results are virtually the same regardless of the starting parameter vector. It is interesting to notice that range-based models (RGARCH and LVE-RGARCH) provide less precise paremeter estimates (higher standard deviations of estimates) than their return-based counterparts. However they are closer to the real values of volatility (they have a lower MAPE).

## 4. Data and results

Daily data including open, high, low and close prices are used. The data set is obtained from the financial website stooq.pl and it covers the period from 1 January 2008 to 31 December 2014. Six assets are examined: EUR/PLN, EUR/USD and EUR/GBP spot rates as well as the Warsaw Stock Exchange WIG20, Dow Jones industrial and DAX indexes. Logarithmic returns are analysed. Logarithmic returns are expressed in percentage points (raw logarithmic returns are multiplied by 100).

In the first step, in-sample analysis is conducted. Models were estimated for the whole analysed period (from 1 January 2008 to 31 December 2014). Quasi maximum likelihood estimates of parameters of the four aforementioned models for all analysed time series are presented in Table 2. These tables display some evident patterns. Coefficient $\alpha$ estimates tend to differ more than coefficient $\beta$ estimates between log-volatility enhanced and single equation models. In the case of the RGARCH model, coefficent $\varpi$ is mostly insignificant at the 0.05 confidence level, whereas in the case of LVE-RGARCH model, the same coeffcient is significant in all but one assets. In line with our expectations, parameter $\rho$ is negative for all stock exchange indexes, and positive for the EUR/PLN pair, while in the case of EUR/USD and EUR/GBP, parameter $\rho$ is insignificant at the 0.05 confidence level. In all cases parameter $k$ estimates are lower than $-v/2$ which confirms the existence of downward bias for Garman-Klass variance estimator.

Using well-known formulas for returns-based GARCH models, as well as recently derived equation (18), unconditional variances of logarithmic returns may be computed. Results are presented in Table 3. As mentioned before, in the case of RGARCH models it is not possible to calculate unconditional variance using parameter estimates. Computing unconditional variance was not possible in the case of LVE-GARCH and LVE-RGARCH models for EUR/USD pair due to the fact that the sum of parameter $\alpha$ and $\beta$ estimates was larger than 1 for the LVE-GARCH model, while in the case of the LVE-RGARCH model condition (19.2) did not hold. However, it should be noticed that even for the standard GARCH model the sum of paremeter $\alpha$ and $\beta$ estimates is very close to 1 indicating that returns process may not be covariance stationary.

Though several diagnostic tests for GARCH-type models can be conducted, two of them are mainly popular. These are tests for the autocorrelation of squared, standardized residuals, and the normality of standardized residuals. Their results are presented in Table 4. In most cases models seem to deal with volatility clustering phenomena, as p-values of Ljung-Box test are greater than 0.05 (with the exception of the RGARCH model for WIG20 as well as the GARCH and LVE-GARCH models for Dow Jones). All models fail to pass the test for normality of standardized residuals.

In the second step of research, out-of-sample analysis is conducted. All four models were estimated on a rolling window of 750 observations. Each time, every model was estimated using the most recent 750 observation, and one-day-ahead volatility forecasts were obtained usign either equation (12) for the GARCH and LVE-GARCH models, or equation (13) for the RGARCH and LVE-RGARCH models. The analysed period again spanned from 1 January 2008 to 31 December 2014, thus the data set was enlarged to include 750 necessary observations prior to 1 January 2008.

A standard way to assess forecasting performance is to calculate forecast error measures, which is straightforward when true values of the forecasted variable are available. If they are not available, one has to use some approximation of the forecasted variable. It is well-known that exact volatility cannot be observed, thus, measures of volatility forecast errors rely heavily on volatility proxy. Patton (2011) thoroughly reviews several forecast error measures and finds that two loss function in particular seem to be more robust to noise in volatility approximations than others. These are the mean squared error and the QLIKE function given by following formula:

$$L(\hat{\sigma}^2, \hat{h}) = \frac{1}{T}\sum_{t=1}^{T} \ln(\hat{h}_t) + \hat{\sigma}_t^2/\hat{h}_t \tag{23}$$

where $\hat{\sigma}_t^2$ is observed volatility on day $t$ measured by some kind of variance estimator (squared returns, range-based estimators, or realized volatility) and $\hat{h}_t$ is a conditional volatility forecast on day $t$.

It should be noticed that QLIKE is an asymmetric loss function. Thus, it tends to favour models that overestimate rather than underestimate true volatility. In this paper, two kinds of observed daily volatility measures are used: the squared daily return and the Garman-Klass estimator.

Both MSE and QLIKE measures were computed for one-day-ahead volatility forecasts obtained from all four analyzed models. Table 5 presents values of aforementioned loss functions (values computed using the Garman-Klass estimator as volatility proxy are marked with asterisk (*)). Range-based models (RGARCH and LVE-RGARCH) seem to outperform their return-based counterparts in the case of currency spot rates. The picture is less clear for the indexes. Pairwise comparison between GARCH and LVE-GARCH, as well as RGARCH and LVE-RGARCH, shows that their values of loss functions are somewhat similar.

Many researchers argue that while forecasting volatility, particular emphasis should be placed on the ability to properly predict tail observations. Thus, an ultimate test of the model's forecasting performance should be computing *Value-at-Risk* and backtesting. In this paper, VaR at 99% level was computed using one-day-ahead volatility predictions from the analyzed models in a following way:

$$VaR_{0.99}(t) = z_{0.01}\sqrt{\hat{h}_t} \tag{24}$$

where $z_{0.01}$ is the first percentile of standard normal distribution, and $\hat{h}_t$ is a conditional volatility forecast on day $t$ obtained using either equation (12) or equation (13).

After counting VaR exceptions, tests for coverage accuracy were conducted: one for unconditional coverage (Kupiec test) and second for conditional coverage (Christoffersen test). The null hypothesis of the Kupiec test is that observed VaR exceptions at level of $(1 - p)\%$, over the period of $n$ days, come from binomial distribution with parameters $p$ and $n$. This test focuses on the frequency of VaR exceptions. The Christoffersen test additionally tests whether VaR exceptions are independent. Full results of backtesting procedure are presented in Table 6. Proposed, log-volatility enhanced models perform much better than their single-equation counterparts. For all but one asset, using bivariate versions of models leads to a lower VaR exception rate, as well as decreases values of tests statistics. The only exception is Dow Jones industrial where RGARCH and its log-volatility enhanced couterpart have the same rate of VaR breaches as well as the same values of tests statistics.

Another interesting question is the behavior of parameter $\rho$ across out-of-sample period. In Figures 1 and 2, point estimates of $\rho$ for both bivariate models are presented for selected assets: the EUR/PLN spot rate and the Dow Jones industrial index. It becomes apparent that in the case of the EUR/PLN spot rate, parameter $\rho$ significantly fluctuates over time. It is less clear in the case of the Dow Jones index where parameter $\rho$ estimates are much more stable. Nevertheless, the assumption of constant correlation between $\varepsilon$ and $\eta$ should be repealed, and formal tests for the stability of the $\rho$ coefficient should be conducted. However, it will be the subject of later research as allowing for time-varying correlation requires reformulation of both the LVE-GARCH and the LVE-RGARCH models. In the analysed graphs reproduced in Figures 1 and 2, one should remember that these are not fitted values of correlation at time $t$, but parameter values obtained from estimating a model on 750 observations prior to $t$.

## 5. Conclusions

In this paper a new approach to modelling volatility is proposed. The main feature of this approach is a dual-equation structure that allows joint distribution of returns and their observed volatility to be modelled. Proposed models may be treated as restricted GARCH (RGARCH) models, where additional equation binds logarithms of observed volatility (measured by extreme value variance estimators) and conditional variance. The proposed framework is very flexible, as it can be modified to incorporate any GARCH-type conditional variance equation. In this paper, equations coming from GARCH and RGARCH (range-based GARCH) models are used. An efficient, range-based Garman-Klass variance estimator is used as an observed volatility approximation. All models are estimated using the time series of the EUR/PLN, EUR/USD and EUR/GBP spot rates, as well as WIG20, Dow Jones industrial and DAX indexes. Pairwise comparison between single-equation and log-volatility enhanced models is conducted. Log-volatility enhanced models do not differ significantly from their single-equation counterparts in terms of forecasting error measures. However, they are much better at coping with *Value-at-Risk* forecasting, resulting in a lower rate of VaR exceptions, as well as lower values of coverage tests statistics.

The approach presented in this paper enables a solution of certain theoretical problem associated with range-based GARCH models to be found. In the single equation RGARCH model, it is not possible to calculate unconditional variance using parameter estimates, because the expected value of the proxy used for volatility is unknown. Incorporating an RGARCH conditional variance equation into proposed bivariate framework solves this problem, as it allows unconditional variance to be determined. It also allows one to infer covariance stationarity of the asset returns process.

Due to the assumed joint normality of returns and logarithms of observed volatility, it is possible to investigate simultaneous dependency between returns and the observed volatility. The findings are in line with expectations: in equity markets the correlation between returns and the observed volatility is negative, and statistically significant, while in the case of foreign exchange markets the sign of correlation differs across the analysed pairs. Empirical results from the out-of-sample analysis indicate that the aforementioned correlation tends to fluctuate; thus, further development of the proposed models is required to incorporate time-varying correlation.

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# Appendix

Table 1

Results of estimating models on 100 artificially generated time series

| | | GARCH | LVE-GARCH | LVE-GARCH* |
|---|---|---|---|---|
| $\varpi \times 10^4$ | mean | 0.1031 | 0.1006 | 0.1006 |
| | st. dev. | 0.024 | 0.009 | 0.009 |
| $\alpha$ | mean | 0.1009 | 0.1001 | 0.1001 |
| | st. dev. | 0.015 | 0.006 | 0.006 |
| $\beta$ | mean | 0.8477 | 0.8492 | 0.8492 |
| | st. dev. | 0.022 | 0.008 | 0.008 |
| MAPE | mean | 3.33% | 2.27% | 2.27% |
| | | **RGARCH** | **LVE-RGARCH** | **LVE-RGARCH*** |
| $\varpi \times 10^4$ | mean | 0.1160 | 0.1043 | 0.1043 |
| | st. dev. | 0.089 | 0.026 | 0.026 |
| $\alpha$ | mean | 0.1066 | 0.1001 | 0.1001 |
| | st. dev. | 0.033 | 0.013 | 0.013 |
| $\beta$ | mean | 0.8357 | 0.8478 | 0.8478 |
| | st. dev. | 0.071 | 0.022 | 0.022 |
| MAPE | mean | 3.34% | 2.25% | 2.25% |

Notes: Estimates of conditional variance parameters are analysed. True parameter values are: $\varpi = 0.00001$, $\alpha = 0.1$, $\beta = 0.85$. The table presents means and standard deviations of parameter estimates, as well as the mean of the MAPE loss function. All are computed using 100 simulated time series of 3000 observations. Models estimated with different values of starting parameters are marked with an asterisk (*).

Table 2
Parameters estimates for analysed time series

| | GARCH | LVE-GARCH | RGARCH | LVE-RGARCH | GARCH | LVE-GARCH | RGARCH | LVE-RGARCH | GARCH | LVE-GARCH | RGARCH | LVE-RGARCH |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **EUR/PLN** | | | | **EUR/USD** | | | | **EUR/GBP** | | | |
| $\mu$ | -0.0094 | 0.0044 | -0.0018 | -0.0017 | -0.0081 | -0.0178 | -0.0160 | -0.0170 | -0.0099 | -0.0018 | -0.0002 | -0.0008 |
| | 0.334 | 0.683 | 0.862 | 0.872 | 0.510 | 0.157 | 0.192 | 0.173 | 0.359 | 0.871 | 0.989 | 0.943 |
| $\varpi$ | **0.0030** | **0.0064** | -0.0021 | **0.0043** | 0.0014 | **0.0011** | 0.0015 | 0.0005 | **0.0022** | **0.0015** | **0.0040** | **0.0019** |
| | 0.019 | 0.000 | 0.324 | 0.005 | 0.243 | 0.014 | 0.548 | 0.622 | 0.031 | 0.000 | 0.043 | 0.014 |
| $\alpha$ | **0.1099** | **0.1226** | **0.2808** | **0.2782** | **0.0405** | **0.0504** | **0.0914** | **0.1139** | **0.0542** | **0.0572** | **0.0923** | **0.1043** |
| | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| $\beta$ | **0.8882** | **0.8669** | **0.7681** | **0.7445** | **0.9564** | **0.9498** | **0.9072** | **0.8914** | **0.9387** | **0.9394** | **0.8888** | **0.8875** |
| | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| $\rho$ | | **0.1392** | | **0.1114** | | -0.0258 | | -0.0187 | | -0.0016 | | -0.0037 |
| | | 0.000 | | 0.000 | | 0.390 | | 0.534 | | 0.956 | | 0.899 |
| $v$ | | **0.5209** | | **0.4857** | | **0.4824** | | **0.4678** | | **0.4595** | | **0.4260** |
| | | 0.000 | | 0.000 | | 0.000 | | 0.000 | | 0.000 | | 0.000 |
| $k$ | | **-0.3970** | | **-0.3518** | | **-0.3062** | | **-0.2745** | | **-0.2240** | | **-0.2047** |
| | | 0.000 | | 0.000 | | 0.000 | | 0.000 | | 0.000 | | 0.000 |
| | **WIG20** | | | | **Dow Jones** | | | | **DAX** | | | |
| $\mu$ | 0.0052 | -0.0268 | -0.0330 | -0.0302 | **0.0649** | 0.0210 | 0.0273 | 0.0299 | **0.0691** | 0.0028 | 0.0098 | 0.0072 |
| | 0.846 | 0.347 | 0.239 | 0.285 | 0.001 | 0.288 | 0.134 | 0.098 | 0.014 | 0.921 | 0.707 | 0.784 |
| $\varpi$ | **0.0156** | **0.0313** | 0.0092 | **0.0290** | **0.0200** | **0.0176** | 0.0013 | **0.0146** | **0.0266** | 0.0181 | **0.0584** | **0.0327** |
| | 0.020 | 0.000 | 0.485 | 0.001 | 0.000 | 0.000 | 0.854 | 0.001 | 0.006 | 0.000 | 0.028 | 0.002 |
| $\alpha$ | **0.0697** | **0.0938** | **0.1872** | **0.2826** | **0.1320** | **0.0956** | **0.4624** | **0.3938** | **0.0849** | **0.0881** | **0.4422** | **0.3850** |
| | 0.000 | 0.000 | 0.024 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| $\beta$ | **0.9242** | **0.8925** | **0.8872** | **0.8225** | **0.8561** | **0.8889** | **0.7109** | **0.7308** | **0.9035** | **0.9073** | **0.6765** | **0.7308** |
| | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| $\rho$ | | **-0.1813** | | **-0.1773** | | **-0.1922** | | **-0.1904** | | **-0.2231** | | **-0.2370** |
| | | 0.000 | | 0.000 | | 0.000 | | 0.000 | | 0.000 | | 0.000 |
| $v$ | | **0.5442** | | **0.5193** | | **0.7032** | | **0.6254** | | **0.6779** | | **0.6034** |
| | | 0.000 | | 0.000 | | 0.000 | | 0.000 | | 0.000 | | 0.000 |
| $k$ | | **-0.8072** | | **-0.8016** | | **-0.7787** | | **-0.6969** | | **-0.7444** | | **-0.6789** |
| | | 0.000 | | 0.000 | | 0.000 | | 0.000 | | 0.000 | | 0.000 |

Notes: Maximum likelihood estimates of model parameters for the analysed time series. Parameters significant at 0.05 level are bolded.

Table 3

Unconditional variances of logarithmic returns of six analysed assets

| | GARCH | LVE-GARCH | RGARCH | LVE-RGARCH |
|---|---|---|---|---|
| EUR/PLN | 1.542 | 0.608 | – | 0.725 |
| EUR/USD | 0.461 | * | – | * |
| EUR/GBP | 0.303 | 0.464 | – | 0.258 |
| WIG20 | 2.580 | 2.293 | – | 2.199 |
| Dow Jones | 1.678 | 1.131 | – | 13.322 |
| DAX | 2.301 | 3.885 | – | 6.385 |

Notes: Unconditional variances of logarithmic returns of six analysed assets for three models (GARCH, LVE-GARCH, LVE-RGARCH). An asterisk (*) means that computing unconditional variance was impossible. Variances are computed for logarithmic returns measured in percentage points.

Table 4

Ljung-Box and Jarque-Bera tests results. In-sample analysis

| | GARCH | LVE-GARCH | RGARCH | LVE-RGARCH |
|---|---|---|---|---|
| | | **EUR/PLN** | | |
| Ljung-Box test | 3.392 | 2.747 | 3.615 | 4.610 |
| p-value | 0.640 | 0.739 | 0.606 | 0.465 |
| Jarque-Bera test | 51.95 | 56.02 | 20.11 | 22.90 |
| p-value | 0.000 | 0.000 | 0.000 | 0.000 |
| | | **EUR/USD** | | |
| Ljung-Box test | 3.542 | 5.493 | 2.416 | 3.139 |
| p-value | 0.617 | 0.359 | 0.789 | 0.679 |
| Jarque-Bera test | 86.36 | 115.33 | 87.59 | 132.12 |
| p-value | 0.000 | 0.000 | 0.000 | 0.000 |
| | | **EUR/GBP** | | |
| Ljung-Box test | 5.973 | 5.006 | 14.548 | 12.985 |
| p-value | 0.309 | 0.415 | 0.012 | 0.024 |
| Jarque-Bera test | 16.59 | 18.68 | 10.31 | 13.85 |
| p-value | 0.000 | 0.000 | 0.006 | 0.001 |
| | | **WIG20** | | |
| Ljung-Box test | 8.152 | 8.019 | 14.960 | 9.746 |
| p-value | 0.148 | 0.155 | 0.011 | 0.083 |
| Jarque-Bera test | 97.04 | 92.43 | 111.94 | 107.56 |
| p-value | 0.000 | 0.000 | 0.000 | 0.000 |
| | | **Dow Jones** | | |
| Ljung-Box test | 15.770 | 23.042 | 7.277 | 9.554 |
| p-value | 0.008 | 0.000 | 0.201 | 0.089 |
| Jarque-Bera test | 83.34 | 92.41 | 39.39 | 39.18 |
| p-value | 0.000 | 0.000 | 0.000 | 0.000 |
| | | **DAX** | | |
| Ljung-Box test | 8.867 | 8.492 | 11.036 | 10.069 |
| p-value | 0.114 | 0.131 | 0.051 | 0.073 |
| Jarque-Bera test | 129.86 | 191.81 | 53.51 | 62.27 |
| p-value | 0.000 | 0.000 | 0.000 | 0.000 |

Notes: Ljung-Box test for autocorrelation of squared, standardized residuals, as well as Jarque-Bera test for normality of standardised residuals with their corresponding p-values computed for all analysed models and assets.

Table 5
Values of loss functions. Out-of-sample analysis

| | GARCH | LVE-GARCH | RGARCH | LVE-RGARCH |
|---|---|---|---|---|
| | | **EUR/PLN** | | |
| MSE | 1.1577 | 1.1634 | 1.1534 | 1.1622 |
| MSE* | 0.4201 | 0.4175 | 0.3819 | 0.3807 |
| QLIKE | -0.2267 | -0.2240 | -0.2574 | -0.2558 |
| QLIKE* | -0.3060 | -0.3048 | -0.3279 | -0.3281 |
| | | **EUR/USD** | | |
| MSE | 0.7219 | 0.7279 | 0.7039 | 0.7042 |
| MSE* | 0.1853 | 0.1901 | 0.1804 | 0.1823 |
| QLIKE | 0.0284 | 0.0126 | -0.0097 | -0.0063 |
| QLIKE* | -0.0035 | -0.0276 | -0.0393 | -0.0349 |
| | | **EUR/GBP** | | |
| MSE | 0.4164 | 0.4193 | 0.4026 | 0.4117 |
| MSE* | 0.1709 | 0.1727 | 0.1540 | 0.1528 |
| QLIKE | -0.3414 | -0.3562 | -0.3636 | -0.3655 |
| QLIKE* | -0.3148 | -0.3244 | -0.3422 | -0.3493 |
| | | **WIG20** | | |
| MSE | 30.9727 | 31.2737 | 31.4498 | 32.1667 |
| MSE* | 8.7210 | 8.5949 | 9.8720 | 10.2803 |
| QLIKE | 1.6121 | 1.6134 | 1.6031 | 1.5988 |
| QLIKE* | 1.2101 | 1.2064 | 1.1939 | 1.1851 |
| | | **Dow Jones** | | |
| MSE | 27.2175 | 27.2068 | 24.6106 | 24.2636 |
| MSE* | 8.8687 | 8.2357 | 9.8644 | 9.2133 |
| QLIKE | 0.9496 | 0.9658 | 0.8821 | 0.8863 |
| QLIKE* | 0.6177 | 0.6219 | 0.5672 | 0.5687 |
| | | **DAX** | | |
| MSE | 39.9396 | 40.1849 | 34.3428 | 35.1679 |
| MSE* | 6.9123 | 7.5221 | 7.2064 | 7.2019 |
| QLIKE | 1.5472 | 1.5511 | 1.4753 | 1.4743 |
| QLIKE* | 1.1862 | 1.1935 | 1.1405 | 1.1444 |

Notes: MSE – mean squared error, QLIKE – "quasi-likelihood" loss function. Loss function values are calculated for conditional volatility predictions in out-of-sample period. Measures with asterisk (*) are computed using Garman-Klass daily variance estimator as a true volatility proxy, measures without asterisk are computed using squared returns.

Table 6

Results of backtesting $VaR_{0.99}$ computed using out-of-sample volatility forecasts

| | GARCH | LVE-GARCH | RGARCH | LVE-RGARCH |
|---|---|---|---|---|
| | | **EUR/PLN** | | |
| % of $VaR_{0.99}$ breaches | 1.615 | 1.392 | 1.615 | 1.392 |
| Unconditional coverage test | 5.779 | 2.484 | 5.779 | 2.484 |
| p-value | 0.0162 | 0.1150 | 0.0162 | 0.1150 |
| Conditional coverage test | 7.186 | 3.932 | 7.186 | 3.932 |
| p-value | 0.0275 | 0.1400 | 0.0275 | 0.1400 |
| | | **EUR/USD** | | |
| % of $VaR_{0.99}$ breaches | 1.761 | 1.211 | 1.101 | 1.046 |
| Unconditional coverage test | 8.668 | 0.764 | 0.180 | 0.038 |
| p-value | 0.0032 | 0.3821 | 0.6711 | 0.8460 |
| Conditional coverage test | 11.990 | 2.291 | 1.797 | 1.710 |
| p-value | 0.0025 | 0.3180 | 0.4071 | 0.4252 |
| | | **EUR/GBP** | | |
| % of $VaR_{0.99}$ breaches | 1.488 | 1.212 | 1.598 | 1.433 |
| Unconditional coverage test | 3.791 | 0.773 | 5.546 | 3.024 |
| p-value | 0.0515 | 0.3794 | 0.0185 | 0.0820 |
| Conditional coverage test | 5.199 | 2.301 | 9.257 | 4.446 |
| p-value | 0.0743 | 0.3165 | 0.0098 | 0.1083 |
| | | **WIG20** | | |
| % of $VaR_{0.99}$ breaches | 1.769 | 1.541 | 1.541 | 1.484 |
| Unconditional coverage test | 8.525 | 4.447 | 4.447 | 3.609 |
| p-value | 0.0035 | 0.0350 | 0.0350 | 0.0575 |
| Conditional coverage test | 19.048 | 12.163 | 21.997 | 16.404 |
| p-value | 0.0001 | 0.0023 | 0.0000 | 0.0003 |

| | | | | |
|---|---|---|---|---|
| **Dow Jones** | | | | |
| % of $VaR_{0.99}$ breaches | 2.155 | 1.929 | 1.758 | 1.758 |
| Unconditional coverage test | 17.865 | 12.074 | 8.355 | 8.355 |
| p-value | 0.0000 | 0.0005 | 0.0038 | 0.0038 |
| Conditional coverage test | 19.473 | 13.559 | 9.791 | 9.791 |
| p-value | 0.0001 | 0.0011 | 0.0075 | 0.0075 |
| **DAX** | | | | |
| % of $VaR_{0.99}$ breaches | 1.854 | 1.742 | 2.191 | 2.022 |
| Unconditional coverage test | 10.474 | 8.096 | 19.036 | 14.500 |
| p-value | 0.0012 | 0.0044 | 0.0000 | 0.0001 |
| Conditional coverage test | 11.926 | 9.521 | 21.715 | 17.404 |
| p-value | 0.0026 | 0.0086 | 0.0000 | 0.0002 |

Note: $VaR_{0.99}$ breaches along with conditional and unconditional coverage tests results.

Figure 1

Parameter $\rho$ estimates for EUR/PLN spot rate across time

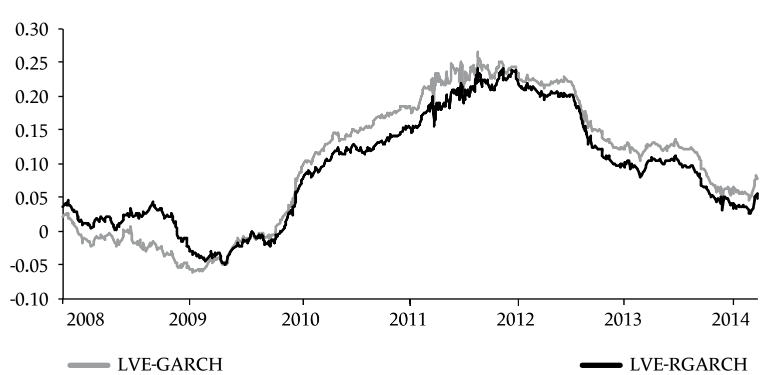

Figure 2
Parameter $\rho$ estimates for Dow Jones industrial index across time